THE STABILITY OF LIQUID FILMS

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Many processes associated with the extraction and concentration of minerals result in the formation of large amounts of foam which must be either dissipated or stabilized. In this connection it is useful to consider as a model the problem of the stability of a free liquid film [1].

1. Let us consider the stability of a layer of viscous incompressible liquid of thickness h with free edges. The solution is found in a cartesian coordinate system whose (x, y) plane coincides with the upper unperturbed surface of the liquid and whose z axis is directed vertically upwards.

The equilibrium state of this system is written in the form

$$\mathbf{v}_i^0 = 0,\ \zeta_i^0 = 0,\ p_i^0 = -\rho_i g z + \mathrm{const}, \tag{1.1}$$

where $\mathbf{v} = \{u, v, w\}$ is the velocity of the liquid; ζ is the displacement of the surface from the equilibrium position; p is pressure; ρ is the density of the liquid; $\mathbf{g} = \{0, 0, -g\}$ is the acceleration of gravity; and i = 1, 2 is the liquid number (a liquid with i = 1 occupies the region $-h \leqslant z \leqslant 0$, and one with i = 2 the region $z \geqslant 0$ and $z \leqslant -h$).

We will investigate the stability of Eq. (1.1), introducing, as usual, velocity and pressure perturbations. Selecting

$$[\alpha/(\rho_1+\rho_2)g]^{1/2},\ [\alpha/(\rho_1+\rho_2)g^3]^{1/4},\quad [\alpha g/(\rho_1+\rho_2)]^{1/4},\ \text{and}\ [\alpha(\rho_1+\rho_2)g]^{1/2}$$

as units of length, time, velocity and pressure respectively, we write the linearized system of equations of motion in the form [2]

$$\frac{\partial \mathbf{v}_i}{\partial t} = -\frac{1}{\beta_i}\nabla p_i + \frac{1}{A_i}\nabla^2 \mathbf{v}_i,\quad \nabla \mathbf{v}_i = 0,\quad i = 1, 2, \tag{1.2}$$

where $\beta_i \equiv \rho_i(\rho_1 + \rho_2)$; $A_i \equiv \nu_i^{-1}[\alpha^3/g(\rho_1+\rho_2)^3]^{1/4}$; α, ν are the surface tension and kinematic viscosity coefficients; $1/A_2 = 0$.

Assuming that the displacement of the surface from the equilibrium position is small, at the phase boundaries we have the following [2]:

at z = 0

$$\frac{\partial \zeta_1}{\partial t} = w_1 = w_2,\ \frac{\partial^2 w_1}{\partial z^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_1 = 0; \tag{1.3}$$

$$p_1 - p_2 = (\beta_1 - \beta_2)\zeta_1 - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta_1 + \frac{2\beta_1}{A_1}\frac{\partial w_1}{\partial z}; \tag{1.4}$$

at z = −H (H is the dimensionless film thickness)

$$\frac{\partial \zeta_2}{\partial t} = w_1 = w_2,\ \frac{\partial^2 w_1}{\partial z^2} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_1 = 0; \tag{1.5}$$

$$p_1 - p_2 = (\beta_1 - \beta_2)\zeta_2 + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta_2 + \frac{2\beta_1}{A_1}\frac{\partial w_1}{\partial z}; \tag{1.6}$$

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as $|z| \to \infty$ $v_2 \to 0$.

Solving the system of equations (1.2) with plane boundary conditions (1.3), (1.5) and the conditions at infinity by the Laplace transform method with respect to time t and the Fourier method with respect to the variables x, y and considering that at the initial instant of time the velocity perturbations and surface displacements are equal to zero and that $\partial v/\partial x$, $\partial v/\partial y$, $\partial \zeta/\partial x$, $\partial \zeta/\partial y \to 0$ for $|x, y| \to \infty$, we obtain the following expressions for the velocity and pressure:

$$W_1(s) = \left(s + \frac{2k^2}{A_1}\right)\frac{[Z_1(s)\,\mathrm{sh}\,k(H+z) - Z_2(s)\,\mathrm{sh}\,kz]}{\mathrm{sh}\,kH} - \frac{2k^2}{A_1}\frac{\left[Z_1(s)\,\mathrm{sh}\sqrt{k^2+A_1s}\,(H+z) - Z_2(s)\,\mathrm{sh}\sqrt{k^2+A_1s}\,z\right]}{\mathrm{sh}\sqrt{k^2+A_1s}\,H},$$

$$W_2(s) = \begin{cases} sZ_1(s)\exp(-kz) & \text{at} \quad z \geqslant 0, \\ sZ_2(s)\exp k(H+z) & \text{at} \quad z \leqslant -H, \end{cases}$$

$$P_1(s) = -s\left(s + \frac{2k^2}{A_1}\right)\frac{\beta_1}{k}\frac{[Z_1(s)\,\mathrm{ch}\,k(H+z) - Z_2(s)\,\mathrm{ch}\,kz]}{\mathrm{sh}\,kH},$$

$$P_2(s) = \begin{cases} \dfrac{s^2\beta_2}{k}Z_1(s)\exp(-kz) & \text{at} \quad z \geqslant 0, \\ -\dfrac{s^2\beta_2}{k}Z_2(s)\exp k(H+z) & \text{at} \quad z \leqslant -H, \end{cases}$$

where W(s), Z(s), P(s) are the Laplace transforms of the quantities w(t), ξ(t), p(t); $k = \{k_x, k_y\}$ is the wave vector; $k^2 \equiv k_x^2 + k_y^2$. Substituting this solution in conditions (1.4), (1.6), we obtain for $Z_i(s)$ the system of equations

$$s^2\beta_2 Z_i(s) + \left(s + \frac{2k^2}{A_1}\right)^2\beta_1\frac{[Z_i(s)\,\mathrm{ch}\,kH - Z_j(s)]}{\mathrm{sh}\,kH} + [k^3 - (-1)^i(\beta_1 - \beta_2)k]\times$$
$$\times Z_i(s) - \beta_1\frac{4k^3}{A_1^2}\sqrt{k^2+A_1s}\,\frac{\left[Z_i(s)\,\mathrm{ch}\sqrt{k^2+A_1s}\,H - Z_j(s)\right]}{\mathrm{sh}\sqrt{k^2+A_1s}\,H} = 0,\ i, j = 1, 2,\ i \neq j. \qquad (1.7)$$

2. Assuming that the viscosity of the liquid is small, i.e. $A_1 \gg 1$, and having carried out the inverse Laplace transformation [3], we obtain for $\zeta_i(t)$, correct to $1/A$, instead of (1.7) the system of equations

$$(\beta_1 + \beta_2\,\mathrm{th}\,kH)\frac{d^2\zeta_i}{dt^2} + 2\delta\beta_1\frac{d\zeta_i}{dt} - \frac{\beta_1}{\mathrm{ch}\,kH}\left(\frac{d^2\zeta_j}{dt^2} + 2\delta\frac{d\zeta_j}{dt}\right) + \zeta_i\Omega_{0i}^2\,\mathrm{th}\,kH = 0, \quad i, j = 1, 2, \quad i \neq j, \qquad (2.1)$$

where $\delta \equiv 2k^2/A_1$; $\Omega_{0i}^2 \equiv k^3 - (-1)^i(\beta_1 - \beta_2)k$.

From these equations it follows that the instability of the liquid film (foam) is caused by a Rayleigh–Taylor instability since $\Omega_{02}^2 < 0$ at $k < 1$. Hence it becomes clear why at high frequencies acoustic or vibro-defoaming is not always effective. Since in this case $\Omega_{0i}^2 = k^3 - (-1)^i(\beta_1 - \beta_2)k + (-1)^i q_i \cos\Omega t$, where q_i and Ω are the dimensionless amplitude and modulation frequency, dynamic stabilization is possible in the system [4, 5].

We seek the solution of system (2.1) in the form $\zeta_i = a_i \exp(\lambda t)$, where a_i are constants. Substituting this solution in the system of equations, we obtain for λ the fourth-order equation

$$\lambda^2\{[\lambda(\beta_1 + \beta_2\,\mathrm{th}\,kH) + 2\delta\beta_1]^2 - \beta_1^2(\lambda + 2\delta)^2/\mathrm{ch}^2kH\} +$$
$$+ \lambda[\lambda(\beta_1 + \beta_2\,\mathrm{th}\,kH) + 2\delta\beta_1](\Omega_{01}^2 + \Omega_{02}^2)\,\mathrm{th}\,kH + \Omega_{01}^2\Omega_{02}^2\,\mathrm{th}^2\,kH = 0. \qquad (2.2)$$

When the film is bordered by a vacuum, i.e., $\beta_2 \equiv 0$, $\beta_1 = 1$, Eq. (2.2) can be simplified and takes the form

$$(\lambda^2 + 2\delta\lambda)^2 + (\lambda^2 + 2\delta\lambda)\frac{(\Omega_{01}^2 + \Omega_{02}^2)\,\mathrm{th}\,kH}{1 - 1/\mathrm{ch}^2kH} + \frac{\Omega_{01}^2\Omega_{02}^2\,\mathrm{th}^2kH}{1 - 1/\mathrm{ch}^2kH} = 0,$$

whence we obtain an expression for λ_*, the increment determining the instability of the film

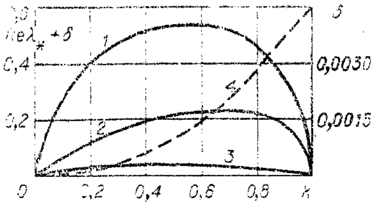

Fig. 1

(foam):

$$\lambda_* = -\delta + \left\{ -\frac{(\Omega_{01}^2 + \Omega_{02}^2)\,\mathrm{sh}\,2kH}{4\,(\mathrm{ch}^2 kH - 1)} + \left[\frac{(\Omega_{01}^2 + \Omega_{02}^2)^2 \mathrm{sh}^2 2kH}{16\,(\mathrm{ch}^2 kH - 1)^2} - \frac{\Omega_{01}^2 \Omega_{02}^2\,\mathrm{sh}^2 kH}{\mathrm{ch}^2 kH - 1} \right]^{1/2} \right\}^{1/2} + o\left(1/A^{3/2}\right). \qquad (2.3)$$

It is not essential to consider the low-viscosity case or that the film is bordered by a vacuum; however, taking arbitrary viscosity into account makes the equations of motion of the film boundaries very clumsy, which complicates the analysis and has practically no effect on the end result (see, e.g., [6]).

Figure 1 shows the dependence of $[\mathrm{Re}\lambda_* + \delta]$ on the wave number of the surface waves excited at the phase boundaries, for various values of the film thickness: curve 1 corresponds to H = 1, curve 2 to H = 0.5, and curve 3 to H = 0.1. When $k = 1$ $\Omega_{02}^2 \equiv k^3 - k = 0$ and $\lambda_* = -\delta$. Since for water when k = 1 $\delta \ll 1$ (curve 4) it can be stated that k = 1 is the stability limit of the film.

From this it follows that only a liquid (water) film of finite dimensions is capable of existing, and in fact the film will be stable if its dimensions do not exceed a value equal to 2π.

If we take into account the effect of the viscosity of the air or a surface-active agent, the energy dissipation increases [7, 8], which leads to an increase in the permissible film dimensions.

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